Calculus II - Day 5

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Comparison Testing

Goals for today:

- Compare series we understand (p-series, geometric) to more complicated series to determine whether they converge or diverge.

Monday: p-test

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

- Converges when $p > 1$
- Diverges when $p \leq 1$

What about these related series?

1. $\sum_{k=1}^{\infty} \frac{1}{k^2+2k}$
2. $\sum_{k=1}^{\infty} \frac{3}{\sqrt{k-\frac{1}{2}}}$

(Direct) Comparison Test:

Let $\sum a_k$ and $\sum b_k$ be two series with positive terms.

1. If $a_k \leq b_k$ for all k and $\sum b_k$ converges, then $\sum a_k$ converges, as well.
2. If $a_k \geq b_k$ for all k and $\sum b_k$ diverges, then $\sum a_k$ diverges, as well.

Ex. $\sum_{k=1}^{\infty} \frac{1}{k^2+2k}$ compare to $\sum_{k=1}^{\infty} \frac{1}{k^2}$

$$a_k = \frac{1}{k^2+2k}, \quad b_k = \frac{1}{k^2}$$

For all k,

$$\frac{1}{k^2+2k} \leq \frac{1}{k^2}$$

and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so by the Comparison Test,

$$\sum_{k=1}^{\infty} \frac{1}{k^2+2k}$$

also converges.

Ex. $\sum_{k=1}^{\infty} \frac{3}{\sqrt{k-1/2}}$ compare to $\sum_{k=1}^{\infty} \frac{3}{\sqrt{k}} = 3\sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$

p-series with $p = 1/2$: diverges.

Since

$$\frac{3}{\sqrt{k-1/2}} \geq \frac{3}{\sqrt{k}}$$

for all positive integers k, by the Comparison Test, our series diverges.

Ex. $\sum_{k=1}^{\infty} \frac{1}{2^k+3^k}$ compare with $\sum_{k=1}^{\infty} \frac{1}{3^k}$

Geometric series with $r = \frac{1}{3}$, so we know it converges.

Since

$$\frac{1}{2^k+3^k} \leq \frac{1}{3^k} \text{ for all } k,$$

$$\sum_{k=1}^{\infty} \frac{1}{2^k+3^k} \text{ converges by the Comparison Test.}$$

Ex. $\sum_{k=2}^{\infty} \frac{3k+6}{\sqrt{k^4-3}}$ compare with $\sum_{k=2}^{\infty} \frac{3}{k}$ (diverges; p-series with $p = 1$)

$$\frac{3k+6}{\sqrt{k^4-3}} \geq \frac{3k}{\sqrt{k^4-3}} \geq \frac{3k}{\sqrt{k^4}} = \frac{3k}{k^2} = \frac{3}{k}$$

Thus,

$$\sum_{k=2}^{\infty} \frac{3k+6}{\sqrt{k^4-3}} \boxed{\text{diverges by the Comparison Test}}$$

Ex. $\sum_{k=2}^{\infty} \frac{\ln(k)}{k}$ let's compare with $\sum_{k=2}^{\infty} \frac{1}{k}$

Is it true that $\frac{\ln(k)}{k} \geq \frac{1}{k}$? ***Not always*** *(not true for $k = 2$)*, but okay when $k \geq 3$. We can still use the Comparison Test, as long as $a_k \geq b_k$ *eventually.* Since $\sum_{k=2}^{\infty} \frac{1}{k}$ diverges and $\frac{1}{k} \leq \frac{\ln(k)}{k}$ eventually,

$$\sum_{k=2}^{\infty} \frac{\ln(k)}{k} \text{ diverges.}$$

Ex. $\sum_{k=2}^{\infty} \frac{5}{k^3+2k+3}$ compare to $\sum_{k=2}^{\infty} \frac{5}{k^3}$

*p-series with p = 3 converges.

Since

$$\frac{5}{k^3+2k+3} \leq \frac{5}{k^3} \text{ for all } k,$$

$\Rightarrow$ our series $\boxed{\text{converges.}}$

Comparison Test: useful when the inequalities "point in the right direction."
Ex. $\sum_{k=2}^{\infty} \frac{5}{k^3-2k+3}$
Issue: $\frac{5}{k^3-2k+3} > \frac{5}{k^3}$ for some k, so we can't directly compare. We need a stronger test...

... The Limit Comparison Test!

Let $\sum a_k$ and $\sum b_k$ be two series with positive terms. Let

$$L = \lim_{k\to\infty} \frac{a_k}{b_k}$$

1. If $0 < L < \infty$, then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.
2. If $L = 0$ ($b_k \gg a_k$), then if $\sum b_k$ converges, then $\sum a_k$ converges.
3. If $L = \infty$ ($b_k \ll a_k$), then if $\sum b_k$ diverges, then $\sum a_k$ diverges.

(We're comparing growth rates.)
Ex. $\sum_{k=2}^{\infty} \frac{5k^4-2k^2+3}{2k^6-k+5}$
Use the Limit Comparison Test (LCT) with $\sum_{k=2}^{\infty} \frac{1}{k^2}$ (which converges).

$$L = \lim_{k\to\infty} \frac{\frac{5k^4-2k^2+3}{2k^6-k+5}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{(5k^4-2k^2+3)k^2}{2k^6-k+5}$$
$$= \frac{5}{2}$$

So by LCT (1), this series converges.

Ex. $\sum_{k=3}^{\infty} \frac{k}{4^k}$ try comparing to $\sum_{k=3}^{\infty} \frac{1}{4^k}$.

$$\lim_{k\to\infty} \left(\frac{\frac{k}{4^k} \cdot 4^k}{\frac{1}{4^k} \cdot 4^k} \right) = \lim_{k\to\infty} \frac{k}{1} = \lim_{k\to\infty} k = \infty$$

$\Rightarrow$ LCT doesn't allow us to make a conclusion from this.

We showed that this series shrinks slower than a series we know converges.
Compare instead with $\sum_{k=3}^{\infty} \frac{1}{2^k}$:

$$\lim_{k\to\infty} \left(\frac{\frac{k}{4^k} \cdot 2^k}{\frac{1}{2^k} \cdot 2^k} \right) = \lim_{k\to\infty} \frac{k \cdot 2^k}{4^k} = \lim_{k\to\infty} \frac{k}{2^k} = 0$$

(Because power grows more slowly than exponentials)

$$\ln(k) \ll k^p \ll b^k \ll k! \ll k^k$$

Ex. $\sum_{k=2}^{\infty} \frac{\ln(k)}{k^2}$

- Won't work if you compare with $\sum \frac{1}{k^2}$ or $\sum \frac{1}{k}$

Try $\frac{1}{k^{1.5}}$:

$$\lim_{k\to\infty} \frac{\left(\frac{\ln(k)}{k^2} \cdot k^{1.5}\right)}{\left(\frac{1}{k^{1.5}} \cdot k^{1.5}\right)} = \lim_{k\to\infty} \frac{\ln(k)}{\sqrt{k}} = 0$$

because $\ln(k) \ll \sqrt{k}$.
Our series is smaller than the convergent series $\sum \frac{1}{k^{1.5}}$, so it $\boxed{\text{converges}}$.